IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING
XLVII CILAMCE
BOGOTÁ - 2026

Linear dynamic stability of a three-story shear building under thermally induced axial compression

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Abstract. This paper investigates the linear dynamic stability of a reduced three-story shear-building model under thermally induced axial compression. The structure is described by three lateral degrees of freedom, equal lumped masses, and equal story stiffnesses. Restrained thermal expansion is represented by an equivalent compressive axial force and combined with gravitational axial forces accumulated along the height. These forces produce a destabilizing geometric stiffness contribution, leading to an effective lateral stiffness matrix. For each prescribed temperature rise, stability is evaluated from the positive definiteness of this matrix and from the associated generalized eigenvalue problem. The critical temperature rise is identified when the minimum eigenvalue of the effective stiffness matrix vanishes, equivalently when the first squared natural frequency becomes zero. A numerical example shows the progressive reduction of the natural frequencies with increasing temperature and gives a critical temperature rise of $916.84°\mathrm{C}$ for the adopted idealized parameters. Phase-plane portraits of the first modal coordinate illustrate the transition from a stable center to an unstable saddle. The result is interpreted as a stability threshold of the reduced linear model, not as a general fire-resistance limit.

Keywords: shear building, dynamic stability, geometric stiffness, thermal load, modal analysis

1 Introduction

Shear-building models are widely used as reduced representations of the lateral dynamic behavior of framed structures. In this class of models, the structure is usually described by lumped masses and equivalent story stiffnesses, which leads to a compact matrix formulation suitable for modal analysis. Aristizabal-Ochoa [1] and Hernandez-Urrea et al. [2] showed that free-vibration properties and stability limits of cantilever shear buildings may be affected by axial loads, base restraints, lumped masses, rotational inertia, and second-order effects associated with lateral displacements. These results support the treatment of axial compression as a destabilizing contribution in simplified structural models.

Thermal loading introduces an additional aspect to this problem. When thermal expansion is restrained, temperature rise may generate compressive stresses that modify the stability condition of structural members. Cai and Feng [3] and Cai et al. [4] showed, for thermally restrained steel columns, that critical temperatures depend on restraint conditions, slenderness, and thermal gradients. Although those studies focus on column buckling, they support the basic mechanism considered here: restrained thermal expansion can produce axial compression and influence stability.

In this work, this mechanism is investigated through a reduced three-story shear-building model. Thermal restraint is represented by an equivalent axial compressive force and combined with gravitational axial forces accumulated along the height. These forces define a geometric stiffness contribution, leading to an effective lateral stiffness matrix. The loss of stability is identified from the loss of positive definiteness of this matrix and, equivalently, from the vanishing of the first squared natural frequency. The objective is not to reproduce the full thermo-mechanical behavior of a real building in fire, but to present a simple and reproducible formulation for linear dynamic stability under thermally induced axial compression.

2 Model formulation

The structure is represented by the three-degree-of-freedom shear-building model shown in Fig. 1. Shear-building idealizations are commonly used in structural dynamics because they provide a compact representation of the lateral motion of framed structures through lumped masses and equivalent story stiffnesses [1, 2]. In the present reduced model, each floor has a lumped mass m, each story has lateral stiffness k, and the generalized coordinates u_1, u_2, and u_3 describe the horizontal floor displacements. The displacement vector and mass matrix are

$$\mathbf{u} = \begin{Bmatrix} u_1 & u_2 & u_3 \end{Bmatrix}^T, \qquad \mathbf{M} = m\mathbf{I}_3, \tag{1}$$

where $\mathbf{I}_3$ is the third-order identity matrix. The interstory drifts are

$$\delta_1 = u_1, \qquad \delta_2 = u_2 - u_1, \qquad \delta_3 = u_3 - u_2. \tag{2}$$

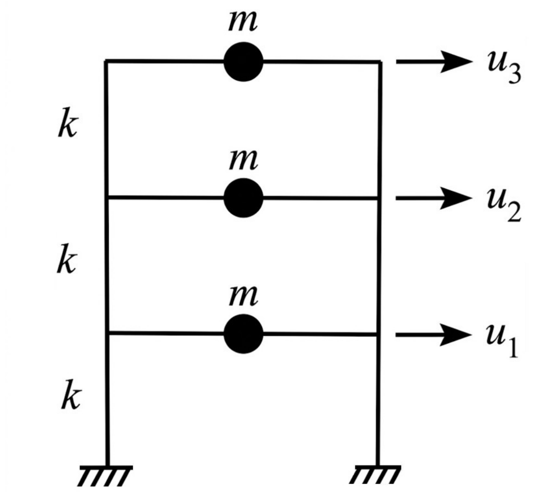


Figure 1. Three-degree-of-freedom shear-building idealization

For equal story stiffnesses, the elastic strain energy is

$$\Pi_e = \frac{1}{2}k\delta_1^2 + \frac{1}{2}k\delta_2^2 + \frac{1}{2}k\delta_3^2 = \frac{1}{2}\mathbf{u}^T\mathbf{K}_e\mathbf{u}, \tag{3}$$

with

$$\mathbf{K}_e = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{4}$$

Compression is taken as positive. In the present reduced model, the destabilizing effect of axial compression is represented through an equivalent geometric stiffness contribution. This representation is consistent with the P–Δ interpretation of axial-load effects discussed for shear-building and beam-column models by Aristizabal-Ochoa [1] and Hernandez-Urrea et al. [2]. Its quadratic potential is written as

$$\Pi_G = -\frac{1}{2}\sum_{i=1}^{3} k_{g,i}\delta_i^2 = \frac{1}{2}\mathbf{u}^T\mathbf{K}_G\mathbf{u}, \tag{5}$$

where

$$\mathbf{K}_G = \begin{bmatrix} -k_{g,1} - k_{g,2} & k_{g,2} & 0 \\ k_{g,2} & -k_{g,2} - k_{g,3} & k_{g,3} \\ 0 & k_{g,3} & -k_{g,3} \end{bmatrix}. \tag{6}$$

The effective lateral stiffness is therefore $\mathbf{K}_{\text{eff}} = \mathbf{K}_e + \mathbf{K}_G$, or

$$\mathbf{K}_{\text{eff}} = \begin{bmatrix} 2k - k_{g,1} - k_{g,2} & -k + k_{g,2} & 0 \\ -k + k_{g,2} & 2k - k_{g,2} - k_{g,3} & -k + k_{g,3} \\ 0 & -k + k_{g,3} & k - k_{g,3} \end{bmatrix}. \tag{7}$$

The geometric stiffness parameters are related to the axial forces by

$$k_{g,i} = \chi \frac{N_i}{h}, \qquad N_i = N_{m,i} + N_{T,i}, \tag{8}$$

where h is the story height and χ is a dimensionless coefficient introduced to represent the participation of the axial force in the equivalent geometric stiffness of the reduced shear-building model. In the present numerical example, $\chi = 1$ is adopted.

The mechanical axial forces represent the cumulative gravitational loads carried by each story:

$$N_{m,1} = 3mg, \qquad N_{m,2} = 2mg, \qquad N_{m,3} = mg, \tag{9}$$

where g is the gravitational acceleration. The thermally induced axial force is written as

$$N_{T,i} = \eta_i E A_{\text{eq}} \alpha \Delta T_i, \tag{10}$$

where E is Young's modulus, A_{eq} is the equivalent axial area of the story, α is the coefficient of thermal expansion, η_i represents the degree of axial restraint, and ΔT_i is the temperature rise. This expression represents the simplified case in which restrained thermal expansion generates an equivalent compressive axial force, a mechanism discussed in thermoelastic buckling analyses by Cai and Feng [3] and Cai et al. [4]. In the numerical analysis, $\Delta T_i = \Delta T$ and $\eta_i = \eta$ for all stories. The resulting P–Δ mechanism is illustrated in Fig. 2.

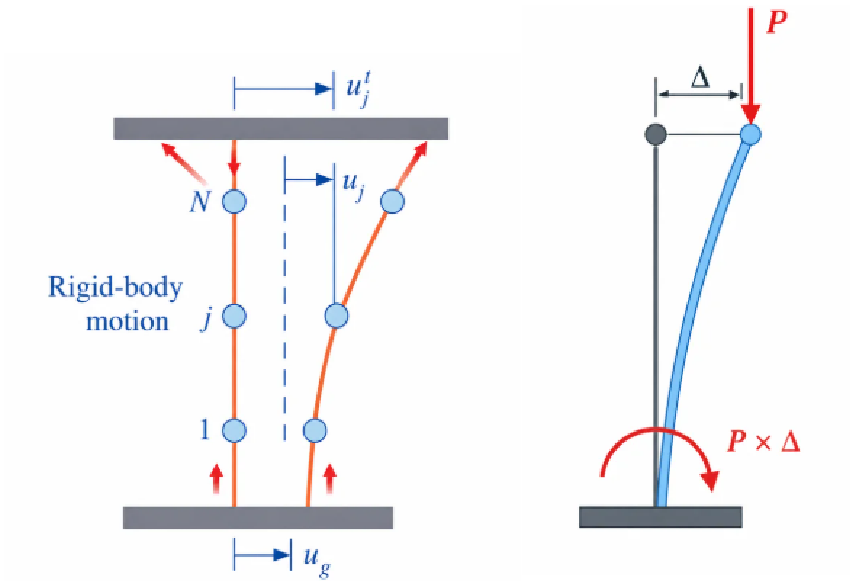


Figure 2. Schematic representation of the P–Δ effect associated with the destabilizing geometric stiffness contribution.

3 Stability criterion

For a fixed value of the temperature rise ΔT, the undamped autonomous equation of motion is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}_{\text{eff}}(\Delta T)\mathbf{u} = \mathbf{0}. \tag{11}$$

The system is conservative for each prescribed value of ΔT, so the total mechanical energy can be written as

$$\mathcal{E} = \frac{1}{2}\dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T \mathbf{K}_{\text{eff}}(\Delta T)\mathbf{u}, \qquad \dot{\mathcal{E}} = 0. \tag{12}$$

Mazzilli et al. [5] and Mazzilli [6] associate the stability of linear conservative systems with the positive definiteness of the quadratic mechanical energy. Energy-based arguments are also discussed in the broader context of structural dynamic stability by Herrmann and Nemat-Nasser [7]. Since $\mathbf{M}$ is positive definite, the stability condition in the present model is governed by the effective stiffness matrix. The critical temperature rise is therefore defined by

$$\Delta T_{\mathrm{cr}} = \min_{\Delta T>0} \left\{\Delta T : \lambda_{\min}\left[\mathbf{K}_{\mathrm{eff}}(\Delta T)\right] = 0\right\}. \tag{13}$$

The equivalent modal criterion follows from the generalized eigenvalue problem

$$\mathbf{K}_{\mathrm{eff}}(\Delta T)\boldsymbol{\phi}_i = \omega_i^2(\Delta T)\mathbf{M}\boldsymbol{\phi}_i. \tag{14}$$

Because $\mathbf{M}$ is positive definite, the first loss of stability also satisfies

$$\omega_1^2(\Delta T_{\mathrm{cr}}) = 0. \tag{15}$$

For mass-normalized modes,

$$\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}, \qquad \boldsymbol{\Phi}^T\mathbf{K}_{\mathrm{eff}}\boldsymbol{\Phi} = \boldsymbol{\Omega}^2, \tag{16}$$

where the columns of $\boldsymbol{\Phi}$ are the mode shapes and $\boldsymbol{\Omega}^2$ is the diagonal matrix of squared natural frequencies [5].

4 Numerical results and discussion

The adopted parameters are

$$\begin{aligned}
&m = 10{,}000\ \mathrm{kg}, \quad k = 10\times10^6\ \mathrm{N/m}, \quad h = 3\ \mathrm{m},\\
&E = 30\ \mathrm{GPa}, \quad A_{\mathrm{eq}} = 0.09\ \mathrm{m}^2, \quad \alpha = 1.2\times10^{-5}\ {}^\circ\mathrm{C}^{-1},\\
&\eta = 1, \quad \chi = 1, \quad g = 9.81\ \mathrm{m/s}^2.
\end{aligned} \tag{17}$$

The temperature rise is assumed uniform over the three stories, with $\Delta T_i = \Delta T$. The parameters E, A_{eq}, and α are kept constant in this linearized example. Solving eq. (13) gives

$$\Delta T_{\mathrm{cr}} \approx 916.84^\circ\mathrm{C}. \tag{18}$$

This value is the critical temperature rise predicted by the reduced linear model. It should not be interpreted as a general fire-resistance limit, since temperature-dependent material degradation, non-uniform heating, and nonlinear effects are not included. These aspects are relevant in more complete thermoelastic buckling analyses, as discussed by Cai and Feng [3] and Cai et al. [4].

At $\Delta T = 0$,

$$\boldsymbol{\omega}(0) = \left\{14.017 \quad 39.316 \quad 56.818\right\}^T\ \mathrm{rad/s}, \tag{19}$$

and, at the critical temperature rise,

$$\boldsymbol{\omega}(\Delta T_{\mathrm{cr}}) = \left\{0 \quad 2.036 \quad 3.934\right\}^T\ \mathrm{rad/s}. \tag{20}$$

Thus, the first mode is the one associated with the first loss of positive definiteness.

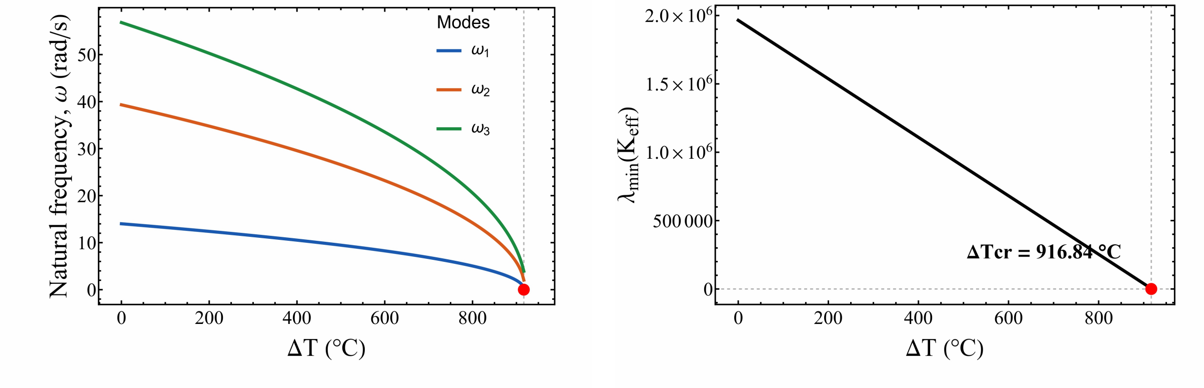
The two plots in Fig. 3 identify the same stability threshold from complementary perspectives. The first natural frequency tends to zero as ΔT approaches ΔT_{cr}, while $\lambda_{\min}(\mathbf{K}_{\mathrm{eff}})$ vanishes at the same point. This agreement confirms the equivalence between the stiffness-definiteness criterion and the modal criterion for the present conservative system.

Near the critical condition, the first modal coordinate satisfies

$$\ddot{q}_1 + \omega_1^2 q_1 = 0. \tag{21}$$

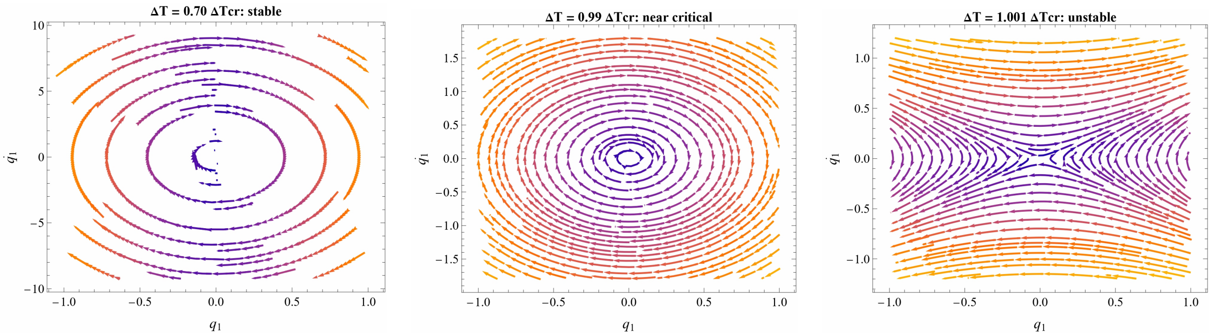
For $\omega_1^2 > 0$, the origin in the $(q_1, \dot{q}_1)$ phase plane is a Lyapunov-stable center, but not an asymptotically stable equilibrium because the system is conservative. Close to the critical condition, the trajectories remain bounded, but the restoring modal stiffness is strongly reduced. For $\omega_1^2 < 0$, writing $\omega_1^2 = -\beta^2$ gives $\ddot{q}_1 - \beta^2 q_1 = 0$, which characterizes a saddle-type instability.

As shown in Fig. 4, the reduced phase portrait of the critical modal coordinate changes qualitatively as the temperature approaches and exceeds the critical value. At $\Delta T = 0.70\Delta T_{\mathrm{cr}}$, the trajectories are closed curves around the origin, indicating a Lyapunov-stable center. At $\Delta T = 0.99\Delta T_{\mathrm{cr}}$, the response is still stable, but the reduced modal stiffness produces slower oscillations. At $\Delta T = 1.001\Delta T_{\mathrm{cr}}$, the change of sign of ω_1^2 transforms the origin into a saddle point, and small perturbations grow along the unstable direction.



(a) (b)

Figure 3. Equivalent identification of the stability threshold: (a) natural frequencies as functions of temperature rise and (b) minimum eigenvalue of $\mathbf{K}_{\text{eff}}$.



(a) (b) (c)

Figure 4. Phase portraits of the first modal coordinate: (a) $\Delta T = 0.70\Delta T_{\text{cr}}$, stable center; (b) $\Delta T = 0.99\Delta T_{\text{cr}}$, near-critical stable response; and (c) $\Delta T = 1.001\Delta T_{\text{cr}}$, unstable saddle.

It should be emphasized that these phase portraits are drawn in the reduced modal plane $(q_1, \dot{q}_1)$, not in the full six-dimensional phase space of the three-degree-of-freedom system.

Table 1. Natural frequencies at selected temperature levels

$\Delta T/\Delta T_{\text{cr}}$	ΔT [°C]	ω_1 [rad/s]	ω_2 [rad/s]	ω_3 [rad/s]
0.00	0.000	14.017	39.316	56.818
0.50	458.421	9.921	27.855	40.259
0.90	825.158	4.468	12.646	18.298
0.99	907.674	1.498	4.548	6.797
1.00	916.843	0.000	2.036	3.934

Table 1 shows the progressive reduction of natural frequencies as the temperature rises. This behavior results from the increasing magnitude of the negative contribution of geometric stiffness produced by thermally induced axial compression. Consequently, the positive-definiteness margin of the effective stiffness matrix decreases with ΔT.

The first mode governs the loss of stability. Its natural frequency decreases from 14.017 rad/s at $\Delta T = 0$ to zero at $\Delta T = \Delta T_{\mathrm{cr}} = 916.843°\mathrm{C}$, so that

$$\omega_1^2(\Delta T_{\mathrm{cr}}) = 0. \tag{22}$$

At the same temperature rise, the second and third frequencies remain positive, with values of 2.036 rad/s and 3.934 rad/s, respectively. Therefore, the first instability is associated with the first vibration mode, while the higher modes still have positive squared natural frequencies at the first critical point.

5 Conclusions

A reduced three-degree-of-freedom shear-building model was formulated from quadratic elastic and geometric potentials to study the effect of thermally induced axial compression on linear dynamic stability. Thermal restraint was represented by an equivalent axial compressive force and combined with gravitational axial forces accumulated along the height. In the numerical example, the critical temperature rise was $\Delta T_{\mathrm{cr}} \approx 916.84°\mathrm{C}$, at which the first natural frequency vanishes and $\mathbf{K}_{\mathrm{eff}}$ loses positive definiteness. Modal and phase-plane analyses showed that the instability is governed by the first mode, whose reduced dynamics changes from a stable center to an unstable saddle after the critical temperature rise. The formulation provides a compact stability analysis for the reduced model adopted. It does not describe post-critical behavior, non-uniform heating, or temperature-dependent material degradation. Therefore, the calculated critical temperature rise should be interpreted as a stability threshold of the idealized linear model, not as a general fire-resistance limit.

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